Indian Statistical Institute
Mid-Semestral Examination
Differential Topology - MMath II

Max Marks: 40 Time: 120 minutes.

Give proper and complete justification(s) for your answers. Throughout $X, Y, \ldots$ are manifolds in some ambient euclidean space.

(1) Decide whether the following statements are correct.
  (a) If X is connected and $f : X \longrightarrow X$ smooth with $f \circ f = f$, then $f(X)$ is a submanifold of X.
  (b) The set $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \ : \ \sum_i x_i^3 = 1, \sum_i x_i = 0\}$ is a manifold.
  (c) Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined by $f(x, y, z) = (xy, yz)$. Then f is transversal to S^1.
  (d) The map $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by
$$t \mapsto \left(\frac{e^t + e^{-t}}{2}, \frac{e^t - e^{-t}}{2} \right)$$
  is an embedding. [10]

(2) Construct a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ such that
$$\text{image}(f) = \{(x, |x|) \ : \ x \in \mathbb{R}\}.$$
[10]

(3) Let $X_1, X_2 \subseteq \mathbb{R}^6$ be the subsets defined by the equations
$$x_1^2 + x_2^2 + x_3^2 - x_4^2 = 1$$
and
$$x_4^2 - x_5^2 - x_6^2 = -1$$
respectively. Show that X_1 and X_2 are manifolds. Do these manifolds intersect transversally? Justify. [2+8]

(4) Show that the map $G : \mathbb{R}^2 \longrightarrow S^1 \times S^1$ defined by $G = g \times g$, where $g : \mathbb{R} \longrightarrow S^1$ is defined as $g(t) = (\cos 2\pi t, \sin 2\pi t)$, is a local diffeomorphism. Further show that if L is a line in $\mathbb{R}^2$, then $G : L \longrightarrow S^1 \times S^1$ is an immersion and if L has irrational slope, then G is one-one on L. [10]